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Point availability of Gaver's duplex system supervised by a safety device unit

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ABSTRACT

We analyse the point availability of Gaver's parallel system supervised by a safety device. For safety reasons, no unit is allowed to operate without supervision. The entire system is attended by two heterogeneous repairmen. Our methodology is based on the theory of sectionally holomorphic functions combined with the notion of dual transforms. As an application we consider Coxian repair time distributions.

1. Introduction

This article continues the analysis of Gaver's parallel system introduced in an earlier article [7]. The version of Gaver's duplex system introduced in [7], termed the S-system, consists of two units (G-units) supervised by a single repairman attending to one failed item at a time. In addition, it contains a governing device (s-unit) whose failure causes the individual G-units to be put into a dormant state (if not already failed) until the s-unit is repaired by a dedicated s-repairman. On the other hand, the s-unit is considered to be dormant if both G-units have failed. Repair times $\mathbf{r}$ of a G-unit and $\mathbf{r_s}$ of the s-unit are assumed to be drawn from general, independent distributions.

The system is represented by a univariate process $\{N_t, t \geq 0\}$ with 5 states:

$N_t = A$: The S-system is fully up,

$N_t = B$: The G-system is dormant, (two operational, but dormant G-units due to an s-unit failure),

$N_t = C$: The S-system is partially up (one G-unit up, one down, and the s-unit is up),

$N_t = C_s$: The G-system is down (both G-units down, one under repair, the s-unit is dormant),

$N_t = D_s$: A failed G-unit and the s-unit are jointly under repair.

For $K = A, B, C, C_s, D_s$ let $p_K(t) := Pr\{N_t = K\}, t \geq 0$, where $\sum_K p_K(t) = 1$. and $N_0 = A$, a.s.

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The purpose of the analysis is to find the point availability of the system $\mathcal{A}(t) \equiv p_A(t) + p_C(t), t \geq 0$, being the probability that the system is up (hence available). First, we use the general birth and death technique, cf. [6] to derive a system of partial differential equations (PDE) with regard to $p_K(t)$. Applying a Laplace–Fourier transform to the PDE leads to a system functional equations. In order to find $\mathcal{A}(t)$, a Sokhotski–Plemelj boundary value problem is constructed from the functional equations and then solved in accordance with the classical Sokhotski–Plemelj Theorem [4]. According to the Theorem the solution is represented by the Cauchy integral. Therefore, in case of a Coxian distribution a closed form solution is obtainable by means of the Residue Theorem and the Inversion Theorem.

2. Assumptions and definitions

2.1. Assumptions

Consider the S-system satisfying the following assumptions. Each operative G-unit has a constant failure rate λ and a general repair time $\mathbf{r}$ with finite mean and distribution $R(\cdot)$, $R(0) = 0$. The operative s-unit has a constant failure rate λ_s and a general repair time $\mathbf{r}_s$ with finite mean and distribution $R_s(\cdot)$, $R_s(0) = 0$. All random variables involved are assumed to be independent and any repair is perfect restoring each unit as-good-as new.

2.2. Definitions

The subsection introduces definitions required for rigorous mathematical justification of the procedure outlined in Section 1.

Characteristic functions and their duals are formulated in terms of a complex transform variable.For instance,

$$\mathbf{E}e^{i\omega\mathbf{r}} = \int_0^\infty e^{iwx}dR(x),\ \operatorname{Im} w \geq 0. \tag{2.1}$$

Note that $\mathbf{E}e^{-i\omega\mathbf{r}} = \int_{-\infty}^0 e^{iwx}d\{1-R((-x)-)\}$, $\operatorname{Im} w \leq 0$. The corresponding Fourier–Stieltjes transforms are called *dual* transforms. Without loss of generality (see Remark 4.1), we may assume that R and R_s have density functions of bounded variation on $[0, \infty)$ with finite mean.

The remaining repair time of the G-unit (the S-unit) being under repair at time t is denoted by X_t (respect. Y_t). The indicator (function) of an event $\{N_t = K\}$ is denoted by $\mathbf{1}\{N_t = K\}$. The complex plane and the real line are respectively denoted by $\mathbf{C}$ and $\mathbf{R}$ with superscript notations such as $\mathbf{C}^+$ and $\mathbf{C}^-$. For instance, $\mathbf{C}^+ := \{w \in \mathbf{C} : \operatorname{Im} w > 0\}$ and $\mathbf{C}^- := \{w \in \mathbf{C} : \operatorname{Im} w < 0\}$. The Laplace transform of any locally integrable and bounded function on $[0, \infty)$ is denoted by the corresponding character marked with an asterisk. For instance,

$$p^*(z) := \int_0^\infty e^{-zt}p(t)dt,\ \operatorname{Re} z > 0.$$

Let $\alpha(\tau)$, $\tau \in \mathbf{R}$ be a bounded and continuous function. $\alpha(\cdot)$ is called Γ-integrable if

$$\lim_{\substack{T\to\infty\\ \varepsilon\downarrow 0}} \int_{\Gamma_{T,\varepsilon}} \alpha(\tau)\frac{d\tau}{\tau-u},\ u \in \mathbf{R}$$

exists, where $\Gamma_{T,\varepsilon} := (-T, u-\varepsilon]\bigcup[u+\varepsilon, T)$. The corresponding integral, denoted by

$$\frac{1}{2\pi i}\int_\Gamma \alpha(\tau)\frac{d\tau}{\tau-u},$$

is called a Cauchy principal value in double sense.

A function $\alpha(\tau)$, $\tau \in \mathbf{R}$ is Lipschitz-continuous (L-continuous) on $\mathbf{R}$ if $\forall\tau_1, \tau_2 \in \mathbf{R}$ there exists a constant c such that $|\alpha(\tau_2)-\alpha(\tau_1)| \leq c|\tau_2-\tau_1|$. The function $\alpha(\tau)$, $\tau \in \mathbf{R}$ is called L-continuous at infinity if $|\alpha(\tau)| = O\left(\frac{1}{|\tau|}\right)$, $|\tau| \to \infty$.

3. Functional equation

In this section we use the general birth and death technique to derive a system of PDEs with regard to p_K, $K = A, B, C, C_s, D_s$. Applying a Laplace–Fourier transform to the PDEs yields a system of functional equations. Finally, we verify that the conditions of the Sokhotski–Plemelj Theorem hold. Let us introduce the measures

$$p_B(t,y)dy := Pr\{N_t = B, y < Y_t \leq y+dy\},$$
$$p_C(t,x)dx := Pr\{N_t = C, x < X_t \leq x+dx\},$$
$$p_{C_s}(t,x)dx := Pr\{N_t = C_s, x < X_t \leq x+dx\},$$
$$p_{D_s}(t,x,y)dxdy := Pr\{N_t = D_s, x < X_t \leq x+dx,\ y < Y_t \leq y+dy\}.$$

Note that, for instance, $p_{D_s}(t) = \int_0^\infty\int_0^\infty p_{D_s}(t,x,y)dxdy$. A general birth and death technique, cf. [6] applied to $\{N_t\}$ yields the set of PDE with initial condition $p_A(0) = 1$.

$$(2\lambda+\lambda_s+\frac{d}{dt})p_A(t) = p_B(t,0)+p_C(t,0),$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial y}\right)p_B(t,y) = \lambda_s p_A(t)\frac{d}{dy}R_s(y)+p_{D_s}(t,0,y),$$

$$\left(\lambda+\lambda_s+\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)p_C(t,x) = (2\lambda p_A(t)+p_{C_s}(t,0))\frac{d}{dx}R(x)+p_{D_s}(t,x,0),$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)p_{C_s}(t,x) = \lambda p_C(t,x),$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)p_{D_s}(t,x,y) = \lambda_s p_C(t,x)\frac{d}{dy}R_s(y),$$

In order to derive $\mathcal{A}^*(z)$, we need a functional equation obtained by a suitable integral transformation of the PDE. For instance, the bivariate transform

$$\int_0^\infty e^{-zt}\mathbf{E}(e^{iwX_t}\mathbf{1}\{N_t = C\})dt,\ \operatorname{Re} z \geq 0, \operatorname{Im} w \geq 0,$$

called a Laplace–Fourier transform. It should be noted that our notation is indispensable to understand the close connection of "dual" transforms in relation with the notion of sectional analytic functions. See, for instance, [5, page 69], Section 4].

Applying a Laplace–Fourier transform technique to the set of differential equations yields the functional equation

$$\begin{aligned}&\left(z+2\lambda(1-\mathbf{E}e^{iw\mathbf{r}})+\lambda_s(1-\mathbf{E}e^{i\eta\mathbf{r_s}})\right)p_A^*(z)-\lambda\psi_C^*(z)\mathbf{E}e^{iw\mathbf{r}}\\&\quad+(z+i\eta)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\eta Y_t}\mathbf{1}\{N_t = B\})dt\\&\quad+\left(z+iw+\lambda+\lambda_s(1-\mathbf{E}e^{i\eta\mathbf{r_s}})\right)\int_0^\infty e^{-zt}\mathbf{E}(e^{iwX_t}\mathbf{1}\{N_t = C\})dt\\&+(z+iw+i\eta)\int_0^\infty e^{-zt}\mathbf{E}(e^{iwX_t}e^{i\eta Y_t}\mathbf{1}\{N_t = D_s\})dt = 1,\end{aligned} \tag{3.1}$$

valid for $\operatorname{Re} z > 0$, $\operatorname{Im} w \geq 0$, $\operatorname{Im}\eta \geq 0$, where

$$\psi_C^*(z) := \int_0^\infty e^{-zt}\mathbf{E}(e^{-zX_t}\mathbf{1}\{N_t = C\})dt.$$

Substituting $w = 0$, $\eta = iz$ into Eq. (3.1) yields the relation

$$(z+\lambda_s(1-\mathbf{E}e^{-z\mathbf{r_s}}))p_A^*(z)+(z+\lambda+\lambda_s(1-\mathbf{E}e^{-z\mathbf{r_s}}))p_C^*(z)-\lambda\psi_C^*(z) = 1. \tag{3.2}$$

We recall that $\mathcal{A}^*(z) = p_A^*(z)+p_C^*(z)$. Thus, in order to derive $\mathcal{A}^*(z)$, we still need two additional independent relations between $p_A^*(z)$, $p_C^*(z)$ and $\psi_C^*(z)$. Therefore, we substitute $w = \tau$, $\eta = -\tau+iz$, $\tau \in \mathbf{R}$, $\operatorname{Re} z > 0$ into the functional equation (3.1). We obtain

$$\begin{aligned}&(z+\lambda_s(1-\mathbf{E}e^{-i(\tau-iz)\mathbf{r_s}})+2\lambda(1-\mathbf{E}e^{i\tau\mathbf{r}}))p_A^*(z)-\lambda\psi_C^*(z)\mathbf{E}e^{i\tau\mathbf{r}}\\&\quad+\gamma^-(\tau,z)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\tau X_t}\mathbf{1}\{N_t = C\})dt-i\tau\int_0^\infty e^{-zt}\\&\quad\times\mathbf{E}(e^{-i(\tau-iz)Y_t}\mathbf{1}\{N_t = B\})dt = 1,\end{aligned} \tag{3.3}$$

where $\gamma^-(\tau,z) := z+i\tau+\lambda+\lambda_s(1-\mathbf{E}e^{-i(\tau-iz)\mathbf{r_s}})$, $\tau \in \mathbf{R}$, $\operatorname{Re} z \geq 0$.

An application of Rouché's Theorem, e.g. [3, page 143] reveals that the function $\gamma^-(w,z)$, $\operatorname{Im} w \leq 0$ has no zeros in $\mathbf{C}^-\bigcup\mathbf{R}$. Dividing Eq. (3.3) by $\gamma^-(\tau,z)$ and separating functions analytic in $\mathbf{C}^+$ (marked by a plus superscript) from functions analytic in $\mathbf{C}^-$ (marked by a minus superscript), yields the boundary value equation

$$\varphi^+(\tau,z)-\varphi^-(\tau,z) = (\lambda\psi_C^*(z)+2\lambda p_A^*(z))\phi(\tau,z),\ \tau \in \mathrm{R}, \tag{3.4}$$

where

$$\varphi^+(w,z) := \int_0^\infty e^{-zt}\mathbf{E}(e^{iwX_t}\mathbf{1}\{N_t = C\})dt,\ \operatorname{Im} w \geq 0,$$

$$\varphi^-(w,z) := (iw\int_0^\infty e^{-zt}\mathbf{E}(e^{-i(w-iz)Y_t}\mathbf{1}\{N_t=B\})dt$$
$$-(z+\lambda_s(1-\mathbf{E}e^{-i(w-iz)\mathbf{r_s}})+2\lambda)p_A^*(z)+1)/\gamma^-(\tau,z),\ \mathrm{Im}\, w\le 0,$$
$$\phi(\tau,z) := \mathbf{E}e^{i\tau\mathbf{r}}/\gamma^-(\tau,z),\ \tau\in \mathrm{R}.$$

Let us prove that Eq. (3.4) satisfies the conditions of the Sokhotski–Plemelj Theorem.

Property 3.1. *The function $\phi(\tau,z)$, $Re\, z\ge 0$ is L-continuous on **R** and at infinity.*

Proof. First note that the assumption of a finite first moment of $\mathbf{r}$ implies that $\left|\frac{\partial}{\partial\tau}\mathbf{E}e^{i\tau\mathbf{r}}\right|\le \mathbf{Er}<\infty$. Applying the Mean Value Theorem for derivatives, e.g. [1], entails that $\left|\mathbf{E}e^{i\tau_2\mathbf{r}}-\mathbf{E}e^{i\tau_1\mathbf{r}}\right|\le \mathbf{Er}\,|\tau_2-\tau_1|$. Hence, $\mathbf{E}e^{i\tau\mathbf{r}}$ is L-continuous on **R**. In a similar way, the assumption $\mathbf{Er}_s<\infty$ and the boundedness of $1/\gamma^-(\tau,z)$, Re $z\ge 0$ implies the L-continuity of $1/\gamma^-(\tau,z)$. Consequently, the function $\phi(\tau,z)$ being a product of two *bounded* L-continuous functions is also L-continuous on **R**. Finally, the boundedness of $1/\gamma^-(\tau,z)$ also implies the L-continuity of $\phi(\tau,z)$ at infinity. Therefore, the properties needed to satisfy the conditions of the Sokhotski–Plemelj Theorem hold.

4. Derivation of $\mathcal{A}^*(z)$

Applying the Sokhotski–Plemelj Theorem (see the Appendix) sustained by Property 3.1 shows that the solution of Eq. (3.4) is given by

$$\varphi(w,z)=Z^*(z)\frac{1}{2\pi i}\int_\Gamma \phi(\tau,z)\frac{d\tau}{\tau-w},\ w\in\mathbf{C}, \tag{4.1}$$

where $Z^*(z):=\lambda\psi_C^*(z)+2\lambda p_A^*(z)$. Moreover, we have

$$\int_0^\infty e^{-zt}\mathbf{E}(e^{iwX_t}\mathbf{1}\{N_t=C\})dt$$
$$=Z^*(z)\frac{1}{2\pi i}\int_\Gamma \phi(\tau,z)\frac{d\tau}{\tau-w},\ w\in\mathbf{C}^+. \tag{4.2}$$

By continuity,

$$\lim_{w\to 0}\int_0^\infty e^{-zt}\mathbf{E}(e^{iwX_t}\mathbf{1}\{N_t=C\})dt=p_C^*(z),$$

whereas by the Sokhotski–Plemelj formulas

$$\lim_{\substack{w\to 0\\ w\in\mathbf{C}^+}}\frac{1}{2\pi i}\int_\Gamma \phi(\tau,z)\frac{d\tau}{\tau-w}=\frac{1}{2}\phi(0,z)+\frac{1}{2\pi i}\int_\Gamma\phi(\tau,z)\frac{d\tau}{\tau}. \tag{4.3}$$

Hence, $p_C^*(z)$ and $\psi_C^*(z)$ can be represented by the Cauchy-type integrals, i.e.

$$p_C^*(z)=Z^*(z)(\frac{1}{2}\phi(0,z)+\frac{1}{2\pi i}\int_\Gamma\phi(\tau,z)\frac{d\tau}{\tau}). \tag{4.4}$$

Similar to $p_C^*(z)$, taking the continuity of $\phi^+(w,z)$, $w\in\mathbf{C}^+$ into account and noting that iz, Re $z>0\in\mathbf{C}^+$ entails that

$$\psi_C^*(z)=Z^*(z)\frac{1}{2\pi i}\int_\Gamma\phi(\tau,z)\frac{d\tau}{\tau-iz}. \tag{4.5}$$

Therefore, $\mathcal{A}^*(z)$ is determined by Eqs. (3.2), (4.4) and (4.5).

Remarks 4.1. It should be noted that the kernel $\phi(\tau,z)$ preserves all the relevant properties to ensure the existence of the Cauchy-type integrals in (4.1)–(4.5) for *arbitrary* repair time distributions R and R_s with finite first moment. In fact, the L-continuity of $\phi(\tau,z)$ on **R** and at infinity does not depend on the canonical structure (Lebesgue decomposition) of the underlying distribution. For instance, the Lipschitz-inequality $|\mathbf{E}e^{i\tau_2\mathbf{r}}-\mathbf{E}e^{i\tau_1\mathbf{r}}|\le \mathbf{Er}|\tau_2-\tau_1|$ always holds for any R with finite mean **Er**. Also the properties of $\gamma^-(\tau,z)$ are preserved. Consequently, our initial assumption concerning the existence of repair time density functions is totally superfluous to ensure the existence of $\mathcal{A}^*(z)$, Re $z>0$.

5. Application example. Coxian distributions

Note that an explicit evaluation of the Cauchy integral as a finite sum of elementary or/and transcendental functions is in general only possible if at least one of the repair time distributions is a Coxian distribution, i.e. a distribution with Laplace–Stieltjes transform of the form $A_m(z)/B_n(z)$, $0\le m<n$, Re $z\ge 0$, where $A_m(z)$ and $B_n(z)$ are polynomials of degree m and n. A suitable model of repair times is the Erlang-K distribution $E_{K,\theta}(u)=1-e^{-\theta u}\sum_{k=0}^{K-1}\frac{(\theta u)^k}{k!}$, $K\ge 1$, being well-known in reliability theory, e.g. [2]. Initially, let $R(\cdot)=e^\mu$ and $R_s(\cdot)$ be arbitrary with finite mean. Eq. (2.1) yields $\mathbf{E}e^{i\tau r}=i\mu(\tau+i\mu)^{-1}$. Invoking the Residue Theorem, e.g. [1,page 468], reveals that

$$\lim_{\substack{w\to 0\\ w\in\mathbf{C}^+}}\frac{1}{2\pi i}\int_\Gamma\phi(\tau,z)\frac{d\tau}{\tau-w}=\frac{1}{z+\mu+\lambda+\lambda_s(1-\mathbf{E}e^{-(z+\mu)\mathbf{r_s}})}, \tag{5.1}$$

whereas

$$\frac{1}{2\pi i}\int_\Gamma\phi(\tau,z)\frac{d\tau}{\tau-iz}=\frac{\mu}{\mu+z}\frac{1}{z+\mu+\lambda+\lambda_s(1-\mathbf{E}e^{-(z+\mu)r_s})}. \tag{5.2}$$

Taking Eqs. (5.1)–(5.2) into account yields the relation

$$\psi_C^*(z)=\frac{\mu}{\mu+z}p_C^*(z). \tag{5.3}$$

Combining Eqs (4.4), (5.1), Eq. (5.3) and the definition of $Z^*(z)$ yields

$$2\lambda p_A^*(z)=p_C^*(z)(z+\mu+\lambda(1-\frac{\mu}{\mu+z})+\lambda_s(1-\mathbf{E}e^{-(z+\mu)\mathbf{r_s}})), \tag{5.4}$$

whereas Eq. (3.2) entails that

$$(z+\lambda_s(1-\mathbf{E}e^{-zr_s}))p_A^*(z)+p_C^*(z)(z+\lambda(1-\frac{\mu}{\mu+z})$$
$$+\lambda_s(1-\mathbf{E}e^{-zr_s}))=1. \tag{5.5}$$

Having determined $p_A^*(z)$ and $p_C^*(z)$ from Eqs. (5.4)–(5.5), we finally obtain $\mathcal{A}^*(z)$. As a numerical example, we focus on the following particular cases:

Case 1: $\lambda=0.1$; $\mu=2$; $\lambda_s=0.5$; $\mu_s=3$.

Let $R_s(\cdot)=E_{2,\mu_s}(\cdot)$. Note that $\mathbf{E}e^{-zr_s}=\mu_s^2/(z+\mu_s)^2=9/(z+3)^2$, whereas $\mathbf{E}e^{-(z+\mu)r_s}=9/(z+5)^2$. Solving the pair of Eqs. (5.4), (5.5) and recalling that $\mathcal{A}^*(z)=p_A^*(z)+p_C^*(z)$, yields

$$\mathcal{A}^*(z)=\frac{N(z)}{zD(z)},\ \mathrm{Re}\, z>0,$$

where

$$N(z)=1252.8+2444.1z+1937.8z^2+798.3z^3+179.8z^4$$
$$+21z^5+z^6,$$

whereas

$$D(z)=1516.5+2857.8z+2170.43z^2+857.02z^3+186.62z^4$$
$$+21.3z^5+z^6.$$

The roots of $D(z)=0$ are

$z_1=-5.30027-1.24164i$; $z_2=\bar{z}_1$; $z_3=-3.24797-1.20131i$, $z_4=\bar{z}_3$, $z_5=-2.48942$, $z_6=-1.71411$.

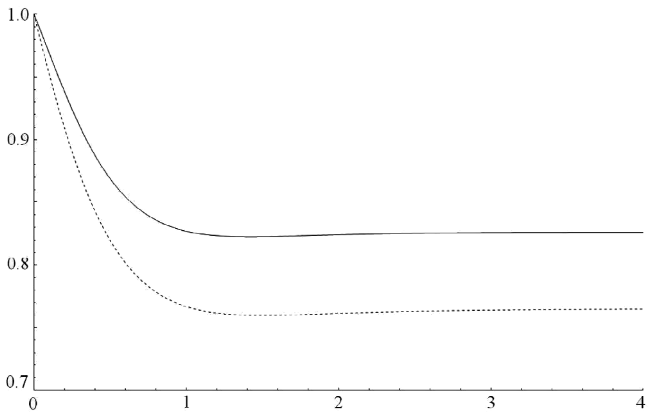

Fig. 5.1. Graph of $\mathcal{A}(t)$, $0 \le t \le 4$. Case 1:solid curve, Case 2:dotted curve.

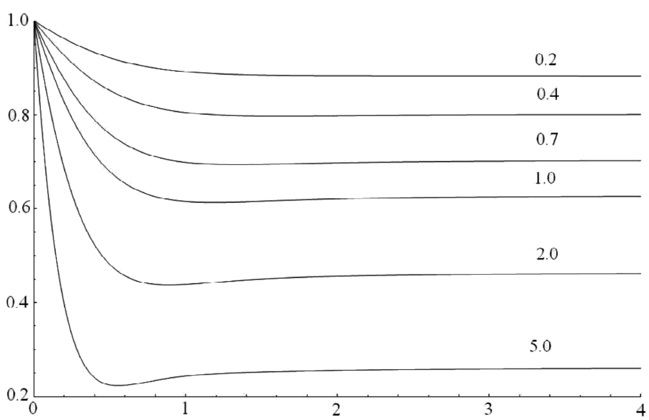

Fig. 5.2. Graph of $\mathcal{A}(t)$, $0 \le t \le 4$, $\lambda_s \in \{0.2; 0.4; 0.7; 1; 2; 5\}$.

Clearly, $\mathcal{A}(t)$ is continuous on $(0, \infty)$ and of bounded variation on $[0, \infty)$. Hence, by the Inversion Theorem

$$\mathcal{A}(t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{-iT+\delta}^{iT+\delta} \frac{e^{zt}N(z)}{z\prod_{k=1}^{6}(z-z_k)} dz, \ \delta > 0, t > 0.$$

Applying the Residue Theorem for Laplace transforms yields

$$\mathcal{A}(t) = 0.81 - 0.07e^{-2.48t} - 0.04e^{-1.71t} - 0.04e^{-5.30t}(\cos 1.24t + \sin 1.24t) + 2e^{-3.24t}(0.17\cos 1.20t + 0.14 \sin 1.20t).$$

Case 2. $\lambda = 0.1$; $\mu = 1$; $\lambda_s = 0.5$; $\mu_s = 3$.

In similar way we obtain

$\mathcal{A}(t) = 0.75 - 0.04e^{-1.37t} - 0.01e^{-0.83t} - 0.04e^{-4.29t}(\cos 0.01t + \sin 0.01) + 2e^{-3.24t}(0.17\cos 1.20t + 0.15\sin 1.20t)$.

Fig. 5.1 shows the graphs of $\mathcal{A}(t)$, $0 \le t \le 4$. Case 1 : solid curve, Case 2 : dashed curve. Finally, we visualize the impact of the safety device on the availability of the S-system by varying the failure rate λ_s. Fig. 5.2 shows the graph of $\mathcal{A}(t)$, $0 \le t \le 4$ for $\lambda_s \in \{0.2; 0.4; 0.7; 1; 2; 5\}$, $\lambda = 0.1$, $\mu = 1$, $\mu_s = 3$.

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Appendix

Let $\alpha(\tau)$ be L-continuous on **R** and infinity. In addition, let

$$\mathcal{L}^+(u) := \lim_{\substack{w\to u\\ w\in \mathbf{C}^+}} \mathcal{L}(w), \quad \mathcal{L}^-(u) := \lim_{\substack{w\to u\\ w\in \mathbf{C}^-}} \mathcal{L}(w), \ u \in \mathbf{R},$$

where

$$\mathcal{L}(w) := \frac{1}{2\pi i}\int_{\Gamma} \alpha(\tau)\frac{d\tau}{\tau - \omega}, \ \omega \in \mathbf{C}.$$

We have

$$\mathcal{L}^+(u) = \frac{1}{2}\alpha(u) + \mathcal{L}(u), \quad \mathcal{L}^-(u) = -\frac{1}{2}\alpha(u) + \mathcal{L}(u).$$

Hence, for $u \in \mathbf{R}$

$$\mathcal{L}^+(u) - \mathcal{L}^-(u) = \alpha(u), \tag{A.1}$$

$$\frac{\mathcal{L}^+(u) + \mathcal{L}^-(u)}{2} = \mathcal{L}(u).$$

Eq. (A.1) is a Sokhotski–Plemelj boundary value problem (on the real line in our case). The intricate problem is to find a sectionally function $\mathcal{L}(w)$, $w \in \mathbf{C}$ having limits $\mathcal{L}^+(u)$, $\mathcal{L}^-(u)$ satisfying Eq. (A.1). See [4] for further details and references to relevant applications.

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